# One Man's Primer for Elementary Calculus: Limits, Continuity and Derivative

James F. Epperson

February 16, 2022

(I originally wrote this document in the early 1980s—before word processing—as a background aid for my Calc II and ODE classes. I post it here to help any students who are insecure regarding their calculus background. I'm open to expanding it if there is enough interest—perhaps include some linear algebra?)

Limit is a mathematical process, an operation. It measures or describes the behavior of a function $f(x)$ as the independent variable, $x$, tends towards a given value $c$.

An intuitive appreciation of this can perhaps be had from the following example. Let $f(n)$ be given by

$$f(n) = 2 - \frac{1}{2^n},$$

for integral values of n, only. (Note that this is a valid function.) The first few values are given by:

$$f(0) = 1, \quad f(1) = \frac{3}{2}, \quad f(2) = \frac{7}{4}, \quad f(3) = \frac{15}{8},$$

and so on. Clearly, the values are tending towards 2 as $n$ gets larger and larger. In fact, we can make $f(n)$ as close to the value 2 as desired, simply by taking $n$ large enough. Note, however, that $f(n)$ never actually reaches 2. Nonetheless, we can say that its limit is 2:

$$\lim_{n\to\infty} f(n) = 2.$$

This is the essence of the limit: that the function values tend towards a value $L$, as the independent variable tends towards a given value $c$. It is important to understand that $f(c)$ itself is unimportant; what truly defines the value of the limit is the behavior of the function for $x$ close to, but not equal to, $c$. Of course, many theorems tell us that we can compute certain limits according to the formula

$$\lim_{x\to c} f(x) = f(c). \tag{1}$$

This is because a large number of functions—though by no means all—fall into the special class of functions known as *continuous* functions. It is only for functions that are continuous at a point $x = c$ that formula (1) applies.

For limit problems not governed by (1), for example,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{2}$$

we must use more subtle methods. (Why does (1) not apply to (2)?) It is virtually impossible to recite such methods in a manner that reduces problems like (2) to formula application, simply because there are too many possible cases. Experience, judgement, and a clear understanding of the concept of limit are the best weapons to use.

Because so much of calculus revolves around functions which are continuous, this concept becomes important enough for brief study in its own right. The definition is contained in (1); this can be broken down into three component parts:

1. $f$ is defined for $x = c$;
2. $\lim_{x \to c} f(x)$ exists and equals $L$;
3. $f(x) = L$.

It is a consequence of the following statement, known as the Intermediate Value Theorem, which forms the basis of our "working man's definition of continuity":

**A function is continuous where its graph can be drawn without lifting the pen from the page.**

Since almost every function that we encounter is continuous nearly everywhere, there is a tendency for students to dismiss the concept as a useless distinction. This is emphatically not the case.

The most practical reason—from a student's point of view—for understanding continuity is contained in equation (1). Since this relationship holds only for continuous functions, and for every continuous function equation (1) is true, then we see that:

Continuous functions are those functions for which we can compute limits easily.

Limits and continuity are studied at different levels by different people. For mathematicians, and those interested in mathematics, these two concepts are immensely important tools for building the foundations of the science. Both ideas can be extended to great generality, and their use and understanding lead to insight into many areas of pure and applied mathematics. For the calculus student, however, the goals are much different. Passing the next test and fulfilling a graduation requirement take precedence over theoretical insight.

This does not diminish the need to either study or learn the material, although the level of expected understanding must be properly set. The student who fails to grasp any of the basic ideas of limit and continuity will find the rest of calculus increasingly difficult, for the concepts do appear again and again, in hypotheses and definitions. The ability to compute limits of contrived functions is not so important for its own sake; it does, however, lend insight into the subtler facets of the limit concept. Similarly, computing the domain of a continuous function is not so special for its own sake, but rather because it tells us when we are dealing with a continuous function.

While the study of limit leads naturally into the study of continuity, it is not the case that continuity leads directly and naturally into the study of derivative. Other motivation must be found.

Perhaps the best approach is to study the geometrical problem of finding the straight line, $\ell(x)$, which is tangent to a given curve $y = f(x)$ at a given point $(c, f(c))$. We find that the slope of this straight line is the numerical value of the derivative.

An alternate interpretation of derivative, which follows from the geometric one discussed above, is rate of change:

The derivative of a function $y = f(x)$ gives the instantaneous rate of change of the dependent variable $y$, with respect to the independent variable $x$.

Thus, a large derivative (high rate of change) means a steep graph; a small derivative (low rate of change) means a shallow graph. The definition of derivative—as distinct from its various interpretations—is given by a limit:

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) \tag{3}$$

Note that the derivative of a function is, itself, another function. Note also that derivatives may not always exist: if the limit in (3) fails to exist for some $x$, then no derivative exists at that point (it may exist elsewhere). Functions for which a derivative exists at a point $c$ are said to be differentiable at that point.

Because of its various interpretations, especially as a rate of change, the derivative is the most important and applicable concept in elementary calculus. There is virtually no limit to what can be done with it. Any problem which involves rates of change or motion implicitly concerns derivatives, and often can be most easily solved using calculus. A large class of scientific and engineering problems can be studied as differential equations, and, in fact, it can plausibly be said that all of our modern technology is based on a thorough understanding of this one mathematical idea.

## Summary of Theorems on Limits and Continuity

Note: Students often do not have a real understanding of what a theorem is, and what it can be used for. The word “theorem” is based on the Greek word for ”truth”; a theorem is therefore a true statement: whenever the hypotheses are true, then the conclusions are true. Always. The fact that something is true, however, does not make it useful in all situations. It is true that $2 + 2 = 4$, but what does that have to do with broiling a steak? Knowing what theorems are available, however is of great help in solving problems, for they can and should be used to reduce a difficult problem to a simple problem. Learning mathematics is, to a great degree, the same as learning when and where to apply what theorem.

1. If $\lim_{x\to c} f(x) = M$ and $\lim_{x\to c} g(x) = L$, then:

    (a) $\lim_{x\to c} (af(x) + bg(x)) = aM + bL$, for $a$ and $b$ constants;
    (b) $\lim_{x\to c}[f(x)g(x)] = ML$;
    (c) $\lim_{x\to c}[f(x)/g(x)] = M/L$, if $L \neq 0$;
    (d) $\lim_{x\to c} (f(x))^{1/n} = M^{1/n}$, if $M > 0$.

2. If $f(x)$ and $g(x)$ are both continuous at a point $x = c$, then:

(a) $(af(x) + bg(x))$ is continuous at $c$, for $a$ and $b$ constants;

(b) $f(x)g(x)$ is continuous at $c$;

(c) $f(x)/g(x)$ is continuous, so long as $g(c) \neq 0$.

(d) $(f(x))^{1/n}$ is continuous, so long as $f(x) > 0$.

3. If $y = f(u)$ is continuous and $u = g(x)$ is continuous, then $h(x) = f(g(x))$ is also continuous.

4. All polynomial functions are continuous everywhere.

5. All rational functions are continuous where they are defined (i.e., where the deno,minator is non-zero).

6. Intermediate Value Theorem: Let $f(x)$ be continuous on the interval $[a, b]$, and let $W$ be a number between $f(a)$ and $f(b)$. Then there exists $\xi \in [a, b]$ such that $f(x) = W$.